

DEGENERATE PULLBACK ATTRACTORS FOR THE 3D NAVIER-STOKES EQUATIONS

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ABSTRACT. As was found in [4], the 3D Navier-Stokes equations with a translationally bounded force contain pullback attractors $\mathcal{A}_w(t)$ in a weak sense. Moreover, those attractors consist of complete bounded trajectories. In this paper, we present a sufficient condition under which the pullback attractors are degenerate. That is, if the Grashof constant is small enough, the pullback attractor will be a single point on a unique, complete, bounded, strong solution. We then apply our results to provide a new proof of the existence of a unique, strong, periodic solution to the 3D Navier-Stokes with a small, periodic forcing term.

1. INTRODUCTION

A natural question in the study of attractors for dissipative partial differential equations is what conditions on the force necessitate a trivial attractor. That is, under what conditions on the force do we find that the attractor $\mathcal{A} = \{z\}$, a single point. This is closely related to the question of dimensionality of the attractor. For the Navier-Stokes equations, it has long been known that they possess a compact global attractor in two dimensions ([9]). The dimension of this global attractor is controlled by the Grashof number $G = \frac{\|f\|_2}{\nu^2 \lambda_1}$ ([6], [7]). In particular, when the Grashof number is small enough, the attractor is trivial. For a proof of this fact, see the book [2], although the argument used goes back to [14]. That is, $\mathcal{A} = \{z\}$ where z is the unique stationary solution to the Stokes system. An analogous result was proven by Chepyzhov and Vishik using trajectory attractors in three dimensions where the Grashof number is given by $G = \frac{\|f\|_2}{\nu^2 \lambda_1^{3/4}}$ ([2]). This result can easily be extended to the theory of weak attractors as developed in [10], [3], [5].

In the nonautonomous Navier-Stokes equations, we have that $f = f(t)$ depends on time. In this situation, we consider the pullback attractor for the system. That is, a family of minimal closed sets $\mathcal{A}(t)$ which uniformly attract all bounded subsets of the phase space in a pullback sense. We will rigorously define these concepts in Section 2.1, below. For more information on the existence and structure of the pullback attractor in two dimensions, we refer the reader to the book [1]. For the existence and structure of pullback attractors for the three dimensional case, we refer the reader to [4]. Now, in the book by Carvalho, Langa, and Robinson [1], they produce a theorem giving sufficient conditions under which the pullback attractor $\mathcal{A}(t)$ for the 2D Navier-Stokes equations is a single point. They find that if a form of the Grashof number is small enough, then the pullback attractor is degenerate.

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That is, if

$$G(t) := \frac{1}{\nu^2 \lambda_1} \left(\limsup_{s \rightarrow -\infty} \frac{1}{t-s} \int_s^t \|f(\xi)\|_2^2 d\xi \right)^{1/2}$$

is small enough, then the pullback attractor $\mathcal{A}(t)$ is trivial. We present an analogous result for the 3D Navier-Stokes equations assuming a translationally bounded force in $L_{loc}^2(\mathbb{R}, L^2)$. We show that if a form of the Grashof number is small enough, then the weak pullback attractor $\mathcal{A}_w(t) = \{v(t)\}$ for a complete, bounded solution v .

The structure of our paper is laid out as follows: Section 2 is devoted to recalling the basic definitions and setup of our problem. We recall the definition of generalized evolutionary systems and pullback attractors as they first appeared in [4]. We then recall the major theorems of existence and structure for the 3D Navier-Stokes equations. Section 3 is then devoted to our proof of the triviality of the pullback attractors under the assumption of small enough Grashof number. We start by proving the strongness of the trajectories on the pullback attractor in Section 3.1. Then, in Section 3.2, we use a modification of Serrin's argument on the uniqueness of strong solutions in intervals of regularity ([18]) to prove that the pullback attractor must be a single point under the usual smallness assumption of the force.

Of particular interest to the reader is Section 3.3. Here, we apply the theorem giving us a unique, bounded, strong solution to the case where the force is periodic. In this setting, we show that this solution is periodic. Theorems proving the existence of periodic solutions to the 3D Navier-Stokes equations go back to Serrin ([17]). Additional results of this type are given in [12], [16], [11] among others. For a more exhaustive discussion of the history of these results, see the recent paper by Kyed ([13]). A common technique in the existence of periodic solutions is the use of Poincaré maps and fixed-point arguments. Instead, we use the structure of the pullback attractor to prove the existence of a periodic solution.

In all that follows, we use the usual conventions of c_0, c_1, \dots for particular (fixed) constants. On the other hand, the constant C will change from line to line.

2. GENERALIZED EVOLUTIONARY SYSTEMS

2.1. Setup and Previous Results. We begin with the setup and definition of a generalized evolutionary system as it appeared in [4]. So, let $(X, d_s(\cdot, \cdot))$ be a metric space with a metric d_s known as the strong metric on X . Let d_w be another metric on X satisfying the following conditions:

1. X is d_w compact.
2. If $d_s(u_n, v_n) \rightarrow 0$ as $n \rightarrow \infty$ for some $u_n, v_n \in X$ then $d_w(u_n, v_n) \rightarrow 0$ as $n \rightarrow \infty$.

As justified by property (2), d_w is called the weak metric on X . For simplicity, denote by X_\bullet the set X with the topology induced by the metric d_\bullet . Next, denote by \overline{A}^\bullet the closure of the set $A \subseteq X$ in the topology generated by d_\bullet . Note that any strongly compact set (d_s -compact) is also weakly compact (d_w -compact), and any weakly closed set (d_w -closed) is also strongly closed (d_s -closed).

Let $C([a, b]; X_\bullet)$, where $\bullet = s$ or w , be the space of d_\bullet -continuous X -valued functions on $[a, b]$ endowed with the metric

$$d_{C([a, b]; X_\bullet)}(u, v) := \sup_{t \in [a, b]} d_\bullet(u(t), v(t)).$$

Let also $C([a, \infty); X_\bullet)$ be the space of all d_\bullet -continuous X -valued functions on $[a, \infty)$ endowed with the metric

$$d_{C([a, \infty); X_\bullet)}(u, v) := \sum_{n \in \mathbb{N}} \frac{1}{2^n} \frac{\sup\{d_\bullet(u(t), v(t)) : a \leq t \leq a + n\}}{1 + \sup\{d_\bullet(u(t), v(t)) : a \leq t \leq a + n\}}.$$

Let

$$\mathcal{T} := \{I \subset \mathbb{R} : I = [T, \infty) \text{ for some } T \in \mathbb{R}\} \cup \{(-\infty, \infty)\},$$

and for each $I \in \mathcal{T}$, let $\mathcal{F}(I)$ denote the set of all X -valued functions on I .

Definition 2.1. A map \mathcal{E} that associates to each $I \in \mathcal{T}$ a subset $\mathcal{E}(I) \subset \mathcal{F}(I)$ will be called a generalized evolutionary system if the following conditions are satisfied:

1. $\mathcal{E}([s, \infty)) \neq \emptyset$ for each $s \in \mathbb{R}$.
2. $\{u(\cdot)|_I : u(\cdot) \in \mathcal{E}(J)\} \subseteq \mathcal{E}(I)$ for each $I, J \in \mathcal{T}$ with $I \subseteq J$.
3. $\mathcal{E}((-\infty, \infty)) = \{u(\cdot) : u(\cdot)|_{[s, \infty)} \in \mathcal{E}([s, \infty)) \ \forall T \in \mathbb{R}\}$.

We refer to $\mathcal{E}(I)$ as the set of all trajectories on the time interval I . Trajectories in $\mathcal{E}((-\infty, \infty))$ are called complete. Next, for each $t \geq s \in \mathbb{R}$ and $A \subseteq X$, define the map

$$P(t, s) : \mathcal{P}(X) \rightarrow \mathcal{P}(X),$$

$$P(t, s)A := \{u(t) : u(s) \in A, u \in \mathcal{E}([s, \infty))\}.$$

This map has the property that, for each $t \geq s \geq r \in \mathbb{R}$ and $A \subseteq X$,

$$P(t, r)A \subseteq P(t, s)P(s, r)A.$$

Other assumptions imposed on \mathcal{E} are given as follows.

- A1 $\mathcal{E}([s, \infty))$ is compact in $C([s, \infty); X_w)$ for each $s \in \mathbb{R}$.
- A2 (Energy Inequality) Let X be a set in some Banach space H satisfying the Radon-Riesz Property (see below) with norm $|\cdot|$ so that $d_s(x, y) = |x - y|$ for each $x, y \in X$, and assume that d_w induces the weak topology on X . Assume that for each $\epsilon > 0$ and each $s \in \mathbb{R}$ there is a $\delta := \delta(\epsilon, s)$ so that for every $u \in \mathcal{E}([s, \infty))$ and $t > s$

$$|u(t)| \leq |u(t_0)| + \epsilon$$

for t_0 a.e. in $(t - \delta, t)$.

- A3 (Strong Convergence a.e.) Let $u, u_n \in \mathcal{E}([s, \infty))$ be so that $u_n \rightarrow u$ in $C([s, t]; X_w)$ for some $s \leq t \in \mathbb{R}$. Then, $u_n(t_0) \xrightarrow{d_s} u(t_0)$ for a.e. $t_0 \in [s, t]$.

Remark 2.2. A Banach space H with norm $|\cdot|$ satisfies the Radon-Riesz property if $x_n \rightarrow x$ in norm if and only if $x_n \rightarrow x$ weakly and

$$\lim_{n \rightarrow \infty} |x_n| = |x|.$$

Often, X will be a closed, bounded subset of a separable, reflexive Banach space. By the Troyanski Renorming Theorem, we can assume that our norm makes H a locally uniformly convex space, at which point the Radon-Riesz property is satisfied.

Definition 2.3. [4] A family of sets $\mathcal{A}_\bullet(t) \subseteq X$ ($t \in \mathbb{R}$) is a d_\bullet -pullback attractor ($\bullet = s$ or w) if $\mathcal{A}_\bullet(t)$ is a minimal set which is

1. d_\bullet -closed.

2. d_\bullet -pullback attracting: for any $B \subseteq X$ and any $\epsilon > 0$, there exists $s_0 := s_0(B, \epsilon) < t \in \mathbb{R}$ so that for $s \leq s_0$,

$$P(t, s)B \subseteq B_\bullet(\mathcal{A}_\bullet(t), \epsilon) := \{u : \inf_{x \in \mathcal{A}_\bullet(t)} d_\bullet(u, x) < \epsilon\}.$$

Definition 2.4. [4] The pullback omega limit set Ω_\bullet ($\bullet = s$ or w) of a set $A \subseteq X$ is a family of sets given by

$$\Omega_\bullet(A, t) := \bigcap_{s \leq t} \overline{\bigcup_{r \leq s} P(t, r)A}.$$

Equivalently, $x \in \Omega_\bullet(A, t)$ if there exist sequences $s_n \rightarrow -\infty$, $s_n \leq t$, and $x_n \in P(t, s_n)A$, such that $x_n \xrightarrow{d_\bullet} x$ as $n \rightarrow \infty$.

Next, we recall the definition of the notion of invariance for a generalized evolutionary system. This requires the following mapping for $A \subseteq X$ and $s \leq t \in \mathbb{R}$:

$$\tilde{P}(t, s)A := \{u(t) : u(s) \in A, u \in \mathcal{E}((-\infty, \infty))\}.$$

Definition 2.5. [4] A family of sets $\mathcal{B}(t) \subseteq X$ is pullback semi-invariant if for each $s \leq t \in \mathbb{R}$,

$$\tilde{P}(t, s)\mathcal{B}(s) \subseteq \mathcal{B}(t).$$

We say that $\mathcal{B}(t)$ is pullback invariant if for each $s \leq t \in \mathbb{R}$,

$$\tilde{P}(t, s)\mathcal{B}(s) = \mathcal{B}(t).$$

$\mathcal{B}(t)$ is pullback quasi-invariant if for each $b \in \mathcal{B}(t)$, there exists a complete trajectory $u \in \mathcal{E}((-\infty, \infty))$ with $u(t) = b$ and $u(s) \in \mathcal{B}(s)$ for all $s \leq t \in \mathbb{R}$.

Definition 2.6. [4] The generalized evolutionary system \mathcal{E} is pullback asymptotically compact if for any $t \in \mathbb{R}$, $s_n \rightarrow -\infty$ with $s_n \leq t$, and any $x_n \in P(t, s_n)X$, the sequence $\{x_k\}$ is relatively strongly compact.

Theorem 2.7. [4] Let \mathcal{E} be a generalized evolutionary system. Then,

1. If the d_\bullet -pullback attractor $\mathcal{A}_\bullet(t)$ exists, then $\mathcal{A}_\bullet(t) = \Omega_\bullet(X, t)$.
2. The weak pullback attractor $\mathcal{A}_w(t)$ exists and is nonempty.

Furthermore, if \mathcal{E} satisfies A1, then

3. $\mathcal{A}_w(t) = \Omega_w(X, t) = \Omega_s(X, t) = \{u(t) : u \in \mathcal{E}((-\infty, \infty))\}$.
4. $\mathcal{A}_w(t)$ is the maximal pullback invariant and maximal pullback quasi-invariant set.
5. (Weak pullback tracking property) For any $\epsilon > 0$ and any $t \in \mathbb{R}$, there exists $s_0 := s_0(\epsilon, t) \leq t$, so that for any $s < s_0$ and $u \in \mathcal{E}([s, \infty))$ satisfies $d_{C([s, \infty); X_w)}(u, v) < \epsilon$, for some complete trajectory $v \in \mathcal{E}((-\infty, \infty))$ with $v(s) \in \mathcal{A}_w(A, s)$ for each $s \leq t$.

Theorem 2.8. [4] Let \mathcal{E} be a pullback asymptotically compact generalized evolutionary system. Then,

1. The strong pullback attractor $\mathcal{A}_s(t)$ exists, it is strongly compact, and $\mathcal{A}_s(t) = \mathcal{A}_w(t)$.

Furthermore, if \mathcal{E} satisfies A1, then

2. (Strong uniform tracking property) For any $\epsilon > 0$ and $t \in \mathbb{R}$ and $T > 0$, there exists $s_0 := s_0(\epsilon, t, T) \leq t$, so that for any $s < s_0$, every trajectory $u \in \mathcal{E}([s, \infty))$ satisfies $d_s(u(t'), v(t')) < \epsilon$ for all $t' \in [s, s + T]$ and some complete trajectory $v \in \mathcal{E}((-\infty, \infty))$ with $v(s) \in \mathcal{A}_s(t)$ for each $s \leq t$.

Theorem 2.9. [4] *Let \mathcal{E} be a generalized evolutionary system satisfying A1, A2, and A3 and so that every complete trajectory is strongly continuous. Then, \mathcal{E} is pullback asymptotically compact.*

2.2. 3D Navier-Stokes Equations. Also in the paper [4], the authors applied the abstract framework of a generalized evolutionary system to the 3D Navier-Stokes equations with a translationally bounded forcing term. Here, we summarize the setup and major results. We will use this setup for the remainder of the paper.

The 3D space-periodic, incompressible Navier-Stokes Equations (NSEs) on the periodic domain $\Omega := \mathbb{T}^3$ are given by

$$(1) \quad \begin{cases} \frac{d}{dt}u - \nu \Delta u + u \cdot \nabla u + \nabla p = f(t) \\ \nabla \cdot u = 0 \end{cases}$$

where u , the velocity vector field, and p , the pressure, are unknowns; $\nu > 0$ is the kinematic viscosity of the fluid, and $f(t) \in L^2_{loc}(\mathbb{R}, H^{-1}(\Omega)^3)$ is a time-dependent forcing term. Assume that the initial condition $u_s := u(s)$ and the forcing term $f(t)$ have the property that

$$\int_{\Omega} u_s(x) dx = \int_{\Omega} f(x, t) dx = 0.$$

Then, we have that

$$\int_{\Omega} u(x, t) dx = 0$$

for all $t \geq s \in \mathbb{R}$.

Let $P_{\sigma} : L^2(\Omega)^3 \rightarrow L^2(\Omega)^3$ be the Leray-Helmholtz projection onto divergence free vector fields (in a distributional sense). Applying this projection to (1), we find that

$$(2) \quad \frac{d}{dt}u - \nu \Delta u + P_{\sigma}(u \cdot \nabla u) = g$$

for $g := P_{\sigma}f$, a differential equation in $H^{-1}(\Omega)^3$.

Note that for notational simplicity, we will write L^2 instead of $L^2(\Omega)^3$ and H^s instead of $H^s(\Omega)^3$ for $s \in \mathbb{R}$. We will also use (\cdot, \cdot) for the L^2 inner product. Other inner products will be labeled appropriately as $(\cdot, \cdot)_H$ for a Hilbert space H . Finally, we use the standard functional pairing between a Banach space X and its dual space X^* as $\langle \cdot, \cdot \rangle$.

Definition 2.10. *The function $u : [s, \infty) \rightarrow L^2$ (or $u : (-\infty, \infty) \rightarrow L^2$) is a weak solution to (2) on $[s, \infty)$ (or $(-\infty, \infty)$) if*

1. $\frac{d}{dt}u \in L^1_{loc}([s, \infty); H^{-1})$.
2. $u \in C([s, \infty); L^2_w) \cap L^2_{loc}([s, \infty); H^1)$.
3. $(\frac{d}{dt}u(t), \phi) - \nu (\Delta u(t), \phi) + (P_{\sigma}(u \cdot \nabla u), \phi) = \langle g(t), \phi \rangle$ for a.e. $t \in [s, \infty)$ and each $\phi \in H^1$.
4. $\nabla \cdot u = 0$ in a distributional sense.

Theorem 2.11 (Leray, Hopf). *For each $u_0 \in L^2$ with $\nabla \cdot u = 0$ (in a distributional sense) and each $g \in L^2_{loc}(\mathbb{R}; H^{-1})$, there exists a weak solution of (2) on $[s, \infty)$ with $u(s) = u_0$, and for each $t \geq t_0$, t_0 a.e. in $[s, \infty)$ we have the following energy inequality:*

$$(3) \quad \|u(t)\|_2^2 + 2\nu \int_{t_0}^t \|\nabla u(s)\|_2^2 ds \leq \|u(t_0)\|_2^2 + 2 \int_{t_0}^t \langle g(s), u(s) \rangle ds.$$

Definition 2.12. A weak solution to (2) satisfying (3) will be called a Leray-Hopf weak solution.

Definition 2.13. A weak solution to (2) satisfying the energy inequality

$$|u(t)|^2 + 2\nu \int_T^t \|u(s)\|^2 ds \leq |u(T)|^2 + 2 \int_T^t \langle g(s), u(s) \rangle ds.$$

for each $t \geq T$ will be called a Leray solution.

Leray solutions are special in that they are continuous at the starting time T . Note that via the Galerkin method, one can prove the existence of Leray solutions for each $u_0 \in H$ and $T \in \mathbb{R}$.

Fix $\tau > 0$. Assume g is translationally bounded in $L_{\text{loc}}^2(\mathbb{R}, H^{-1})$. That is,

$$\|g\|_{L_b^2(\tau)}^2 := \sup_{t \in \mathbb{R}} \frac{1}{\tau} \int_t^{t+\tau} \|g(s)\|_{H^{-1}}^2 ds < \infty.$$

First, note that $\|g\|_{H^{-1}}$ and $\|g\|_{L_b^2(\tau)}$ have the same dimensions. Next, note that the choice of τ is not particularly important. In fact, for any $\tau, \rho > 0$, we have that the norms $\|\cdot\|_{L_b^2(\tau)}$ and $\|\cdot\|_{L_b^2(\rho)}$ are equivalent.

Lemma 2.14. Let $\tau, \rho > 0$ be given. Assume, without loss of generality that $\tau \leq \rho$. Then, for any translationally bounded $g \in L_{\text{loc}}^2(\mathbb{R}, H^{-1})$,

$$\frac{\tau}{\rho} \|g\|_{L_b^2(\tau)}^2 \leq \|g\|_{L_b^2(\rho)}^2 \leq \frac{N\tau}{\rho} \|g\|_{L_b^2(\tau)}^2,$$

where N is any integer so that $N\tau \geq \rho$.

The proof is elementary and is thus omitted. Therefore, we may use whatever $\tau > 0$ we like in our calculations. Later, we will choose $\tau := (\nu\lambda_1)^{-1}$.

As was shown in [4], there exists an absorbing ball for Leray solutions of (2). That is, from the energy inequality and the fact that g is translationally bounded, one can derive the following inequality:

$$\|u(t)\|_2^2 \leq \|u(t_0)\|_2^2 e^{\nu\lambda_1(t_0-t)} + \frac{\tau \|g\|_{L_b^2(\tau)}^2}{\nu(1 - e^{-\nu\lambda_1\tau})}.$$

Letting

$$(4) \quad R := \frac{2\tau \|g\|_{L_b^2(\tau)}^2}{\nu(1 - e^{-\nu\lambda_1\tau})},$$

we define

$$X := \{u \in L^2 : \|u\|_2^2 \leq R\}$$

as a closed absorbing ball in L^2 . In particular, X is weakly compact with strong and weak metrics given by

$$d_s(u, v) := \|u - v\|_2 \quad \text{and} \quad d_w(u, v) := \sum_{k \in \mathbb{Z}^3} \frac{1}{2^{|k|}} \frac{|\hat{u}_k - \hat{v}_k|}{1 + |\hat{u}_k - \hat{v}_k|}$$

for $u, v \in L^2$ where \hat{u}_k and \hat{v}_k are the Fourier coefficients of u and v , respectively. Note that the above weak metric d_w induces the weak topology on X .

Next, we define our generalized evolution system on X by

$$\begin{aligned}\mathcal{E}([s, \infty)) &:= \{u : u \text{ is a Leray-Hopf solution of (2) on } [s, \infty) \\ &\quad \text{and } u(t) \in X \text{ for } t \in [s, \infty)\}, \\ \mathcal{E}((-\infty, \infty)) &:= \{u : u \text{ is a Leray-Hopf solution of (2) on } (-\infty, \infty) \\ &\quad \text{and } u(t) \in X \text{ for } t \in (-\infty, \infty)\}.\end{aligned}$$

Then, \mathcal{E} satisfies the necessary properties in Definition 2.1 and forms a generalized evolutionary system on X . We must use Leray-Hopf weak solutions in the definition of our evolutionary system since the restriction of a Leray solution is not necessarily a Leray solution. However, the restriction of a Leray solution is always a Leray-Hopf weak solution. In fact, \mathcal{E} satisfies A1. Therefore, by Theorem 2.7, we have the following theorem.

Theorem 2.15. [4] *Let g be translationally bounded in $L^2_{loc}(\mathbb{R}, H^{-1})$. Then, there exists a weak pullback attractor $\mathcal{A}_w(t)$ for the generalized evolutionary system \mathcal{E} of Leray-Hopf weak solutions to (2). In particular,*

$$\mathcal{A}_w(t) = \{u(t) : u \in \mathcal{E}((-\infty, \infty))\}$$

is the maximal invariant and quasi-invariant subset of X .

In particular, there exists a complete bounded (in the sense of L^2) weak solution to the 3D Navier-Stokes equations. In the next section, we will present an argument demonstrating that when the force is small enough, the weak pullback attractor consists of only one such solution. In this case, we have that

$$\mathcal{A}_w(t) = \{u(t)\},$$

is trivial.

3. DEGENERATE PULLBACK ATTRACTORS

3.1. A Criterion for Strong Solutions. In our goal of proving the existence of that the pullback attractor consists of a single point, we will begin by showing that if the force is small enough, then a complete bounded solution guaranteed by Theorem 2.15 is, in fact, a strong solution.

Definition 3.1. *A weak solution u to (2) will be called strong if $u \in L^\infty_{loc}(\mathbb{R}, H^1)$.*

Let v be a complete bounded solution to (2) as discussed in the previous section. In particular, v satisfies the inequality (13). Using the Cauchy-Schwarz inequality followed by Young's inequality, we find that

$$(5) \quad \|v(t)\|_2^2 + \nu \int_s^t \|\nabla v(\xi)\|_2^2 d\xi \leq \|v_0\|_2^2 + \frac{1}{\nu} \int_s^t \|g(\xi)\|_{H^{-1}}^2 d\xi.$$

Using the radius of the absorbing ball given in (4) and dropping the first term on the left-hand side, we find that

$$\nu \int_s^t \|\nabla v(\xi)\|_2^2 d\xi \leq \frac{2\tau \|g\|_{L_b^2(\tau)}^2}{\nu(1 - e^{-\nu\lambda_1\tau})} + \frac{1}{\nu} \int_s^t \|g(\xi)\|_{H^{-1}}^2 d\xi.$$

Thus, we find that for any $s \in \mathbb{R}$

$$(6) \quad \int_s^{s+\tau} \|\nabla v(\xi)\|_2^2 d\xi \leq \frac{\tau \|g\|_{L_b^2(\tau)}^2 (3 - e^{-\nu\lambda_1\tau})}{\nu^2 (1 - e^{-\nu\lambda_1\tau})}.$$

Hence, we find that for any $M \geq 0$,

$$|\{x \in [s, s + \tau] : \|\nabla v(x)\|_2 \geq M\}| \leq \frac{1}{M^2} \frac{\tau \|g\|_{L_b^2(\tau)}^2 (3 - e^{-\nu \lambda_1 \tau})}{\nu^2 (1 - e^{-\nu \lambda_1 \tau})}.$$

Letting $M := \left(\frac{2 \|g\|_{L_b^2(\tau)}^2 (3 - e^{-\nu \lambda_1 \tau})}{\nu^2 (1 - e^{-\nu \lambda_1 \tau})} \right)^{-1/2}$, we have that

$$|\{x \in [s, s + \tau] : \|\nabla v(x)\|_2 \geq M\}| \leq \frac{\tau}{2}.$$

We encapsulate the above remarks into the following lemma.

Lemma 3.2. *Let v be any complete, bounded solution to (2) with g translationally bounded in $L_{loc}^2(\mathbb{R}, H^{-1})$ whose existence is guaranteed by Theorem 2.15. Then, for any $t \in \mathbb{R}$, there exists a point $s \in [t, t + \tau]$ so that*

$$\|\nabla v(s)\|_2^2 \leq \frac{2 \|g\|_{L_b^2(\tau)}^2 (3 - e^{-\nu \lambda_1 \tau})}{\nu^2 (1 - e^{-\nu \lambda_1 \tau})} < \infty.$$

Now, we add the assumption that g is translationally bounded in $L_{loc}^2(\mathbb{R}, L^2)$ which will be assumed for the remainder of the paper. That is, we assume that

$$\|g\|_{L_0^2(\tau)}^2 := \sup_{t \in \mathbb{R}} \frac{1}{\tau} \int_t^{t+\tau} \|g(\xi)\|_2^2 d\xi < \infty.$$

Note that using the Poincarè inequality, we have that

$$\|g\|_{L_b^2(\tau)}^2 \leq \lambda_1^{-1} \|g\|_{L_0^2(\tau)}^2.$$

We will show that if $\|g\|_{L_0^2(\tau)}$ is sufficiently small, then $v \in L^\infty(\mathbb{R}, H^1)$.

To do this, let $t_0 \in \mathbb{R}$ be arbitrary. Then, consider the interval $[t_0 - \tau, t_0]$. By Lemma 3.2, there exists a point $s \in [t_0 - \tau, t_0]$ so that

$$\|\nabla v(s)\|_2^2 \leq \frac{2 \|g\|_{L_b^2(\tau)}^2 (3 - e^{-\nu \lambda_1 \tau})}{\nu^2 (1 - e^{-\nu \lambda_1 \tau})} < \infty.$$

Thus, by Leray's characterization [15], there is an $\epsilon > 0$ so that v is a strong solution on $[s, s + \epsilon]$. We investigate the length of this interval.

Starting with (2), we take the inner product with $-\Delta v$ giving us that

$$(7) \quad \frac{1}{2} \frac{d}{dt} \|\nabla v\|_2^2 + \nu \|\Delta v\|_2^2 \leq |(P_\sigma(v \cdot \nabla v), \Delta v)| + |(g, \Delta v)|.$$

Classical estimates give us that

$$(8) \quad |(P_\sigma(v \cdot \nabla v), \Delta v)| \leq c_0 \|\nabla v\|_2^{3/2} \|\Delta v\|_2^{3/2}$$

$$(9) \quad |(g, \Delta v)| \leq \|g\|_2 \|\Delta v\|_2.$$

Next, we apply Young's inequality on each of these terms to get that

$$\begin{aligned} |(P_\sigma(v \cdot \nabla v), \Delta v)| &\leq \frac{\nu}{4} \|\Delta v\|_2^2 + \frac{c_0}{\nu^3} \|\nabla v\|_2^6 \\ |(g, \Delta v)| &\leq \frac{1}{\nu} \|g\|_2^2 + \frac{\nu}{4} \|\Delta v\|_2^2. \end{aligned}$$

Using these estimates as well as the Poincarè inequality, (7) reduces to

$$(10) \quad \frac{d}{dt} \|\nabla v\|_2^2 + \nu \lambda_1 \|\nabla v\|_2^2 \leq \frac{2}{\nu} \|g\|_2^2 + \frac{c_0}{\nu^3} \|\nabla v\|_2^6.$$

Now, assume that

$$\|g\|_{L_b^2(\tau)}^2 \leq \frac{c_0^{-1/2} \nu^4 \lambda_1^{3/2}}{2c_1 + 4\nu\lambda_1\tau}$$

where $c_1 := \frac{2(3-e^{-\nu\lambda_1\tau})}{1-e^{-\nu\lambda_1\tau}}$. Then, we will show that $\|\nabla v(t_0)\|_2^2 \leq c_0^{-1/2} \nu^2 \lambda_1^{1/2}$. The following is a modification of the argument given in [8]. For completeness, we present the argument in its entirety.

First, note that the criterion on $\|g\|_{L_b^2(\tau)}$ guarantees that

$$\begin{aligned} \|\nabla v(s)\|_2^2 + \frac{2}{\nu} \int_s^{s+\tau} \|g(\xi)\|_2^2 d\xi &\leq \frac{c_1}{\nu^2} \|g\|_{L_b^2(\tau)}^2 + \frac{2\tau}{\nu} \|g\|_{L_b^2(\tau)}^2 \\ &\leq \frac{c_1}{\nu^2 \lambda_1} \|g\|_{L_b^2(\tau)}^2 + \frac{2\tau}{\nu} \|g\|_{L_b^2(\tau)}^2 \\ &\leq \frac{c_0^{-1/2} \nu^2 \lambda_1^{1/2}}{2}. \end{aligned}$$

Then, certainly $\|\nabla v(s)\|_2^2 < c_0^{-1/2} \nu^2 \lambda_1^{1/2}$. Let

$$T := \sup\{t \in [s, s+\tau] : \|\nabla v(t)\|_2^2 < c_0^{-1/2} \nu^2 \lambda_1^{1/2}\}.$$

Since v is a strong solution at s we get that $T > s$. Assume that $T < s+\tau$. Using $\|\nabla v(T_0)\|_2^2 < c_0^{-1/2} \nu^2 \lambda_1^{1/2}$ for each $T_0 \leq T$, we find that

$$\nu\lambda_1 \|\nabla v(T_0)\|_2^2 - \frac{c_0}{\nu^3} \|\nabla v(T_0)\|_2^6 = \nu\lambda_1 \|\nabla v(T_0)\|_2^2 \left(1 - \frac{c_0}{\nu^4 \lambda_1} \|\nabla v(T_0)\|_2^4\right) \geq 0.$$

Thus, we integrate (10) from s to T and get that

$$\begin{aligned} \|\nabla v(T)\|_2^2 &\leq \|\nabla v(s)\|_2^2 + \frac{2}{\nu} \int_s^T \|g(\xi)\|_2^2 d\xi \\ &\leq \|\nabla v(s)\|_2^2 + \frac{2}{\nu} \int_s^{s+\tau} \|g(\xi)\|_2^2 d\xi \\ &\leq \frac{c_0^{-1/2} \nu^2 \lambda_1^{1/2}}{2}. \end{aligned}$$

Thus, we must have that $T = s + \tau$. In particular, this is true of $t_0 \in [s, s + \tau]$. Since $t_0 \in \mathbb{R}$ was arbitrary, we have that

$$(11) \quad \|\nabla v(s)\|_2^2 < c_0^{-1/2} \nu^2 \lambda_1^{1/2}$$

for all $s \in \mathbb{R}$. This completes the proof of the following theorem.

Theorem 3.3. *Suppose g is translationally bounded in $L_{loc}^2(\mathbb{R}; L^2)$ so that*

$$\|g\|_{L_b^2(\tau)}^2 \leq \frac{c_0^{-1/2} \nu^4 \lambda_1^{3/2}}{2c_1 + 4\nu\lambda_1\tau}$$

for $c_1 := \frac{2(3-e^{-\nu\lambda_1\tau})}{1-e^{-\nu\lambda_1\tau}}$ and c_0 the constant given in (8). Then, there exists a complete, bounded, strong solution to (2) so that $v \in L^\infty(\mathbb{R}; H^1)$. In particular, $\|\nabla v(s)\|_2^2 < c_0^{-1/2} \nu^2 \lambda_1^{1/2}$ for all $s \in \mathbb{R}$.

Now, let's let $\tau := (\nu\lambda_1)^{-1}$. For simplicity, we let

$$\begin{aligned} \|g\|_{L_b^2((\nu\lambda_1)^{-1})} &:= \|g\|_{L_b^2} \\ \|g\|_{L_b^2((\nu\lambda_1)^{-1})} &:= \|g\|_{L_b^2}. \end{aligned}$$

Then, we can express Theorem 3.3 in terms of the non-dimensional 3D Grashof number

$$G := \frac{\|g\|_{L_0^2}}{\nu^2 \lambda_1^{3/4}}.$$

Corollary 3.4. *Suppose g is translationally bounded in $L_{loc}^2(\mathbb{R}; L^2)$ so that*

$$G^2 = \frac{\|g\|_{L_0^2}^2}{\nu^4 \lambda_1^{3/2}} \leq \frac{c_0^{-1/2}}{2c_1 + 4}$$

for $c_1 := \frac{2(3-e^{-1})}{1-e^{-1}}$ and c_0 the constant given in (8). Then, there exists a complete, bounded, strong solution to (2) so that $v \in L^\infty(\mathbb{R}; H^1)$. In particular, $\|\nabla v(s)\|_2^2 < c_0^{-1/2} \nu^2 \lambda_1^{1/2}$ for all $s \in \mathbb{R}$.

It is also worthwhile to note that the above argument proves the strongness of all complete trajectories in our generalized evolutionary system \mathcal{E} . In fact, it proves that if $u \in \mathcal{E}([s, \infty))$, then for $t > s + \tau$, u is strong.

3.2. A Serrin-type Argument. In ([18]), Serrin presents an argument for the uniqueness of weak solutions in an interval of regularity (where a strong solution exists). Using a modification of the argument as it is presented in ([19]), we obtain the required argument for the existence of degenerate pullback attractors.

Let v be a complete, bounded strong solution to (2) on $(-\infty, \infty)$ guaranteed by Theorem 3.3. Let u be another Leray-Hopf weak solution to (2) on $[T, \infty)$ and let $w := u - v$. Then, u and v satisfy

$$(12) \quad \|u(t)\|_2^2 + 2\nu \int_s^t \|\nabla u(\xi)\|_2^2 d\xi \leq \|u_0\|_2^2 + 2 \int_s^t \langle g(\xi), u(\xi) \rangle d\xi,$$

$$(13) \quad \|v(t)\|_2^2 + 2\nu \int_s^t \|\nabla v(\xi)\|_2^2 d\xi = \|v_0\|_2^2 + 2 \int_s^t \langle g(\xi), v(\xi) \rangle d\xi,$$

respectively for each $t \geq s \geq T$ since v is a strong solution. Also, as seen in Temam's book [19]

$$(14) \quad \begin{aligned} (u(t), v(t)) + 2\nu \int_s^t (u(\xi), v(\xi))_{H^1} d\xi &= (u(s), v(s)) \\ &\quad + \int_s^t \langle g(\xi), u(\xi) + v(\xi) \rangle d\xi \\ &\quad - \int_s^t \langle w(\xi) \cdot \nabla w(\xi), v(\xi) \rangle d\xi. \end{aligned}$$

Adding (12) to (13) and then subtracting twice (14), we get that

$$(15) \quad \|w(t)\|_2^2 + 2\nu \int_s^t \|\nabla w(\xi)\|_2^2 d\xi \leq \|w(s)\|_2^2 + 2 \int_s^t \langle w(\xi) \cdot \nabla w(\xi), v(\xi) \rangle d\xi.$$

We estimate the nonlinear term using classical estimates. That is, we find that

$$\begin{aligned} |\langle w \cdot \nabla w, v \rangle| &\leq C \|w\|_2^{1/4} \|\nabla w\|_2^{7/4} \|v\|_2^{1/4} \|\nabla v\|_2^{3/4} \\ &\leq \frac{\nu}{2} \|\nabla w\|_2^2 + \frac{C}{\nu^7} \|v\|_2^2 \|\nabla v\|_2^6 \|w\|_2^2 \end{aligned}$$

after applying the Young's inequality. Since $v \in L^\infty(\mathbb{R}, H^1) \cap L^\infty(\mathbb{R}, L^2)$, we use (4) and (11) to estimate (15) by

$$(16) \quad \|w(t)\|_2^2 - \|w(s)\|_2^2 \leq \nu \lambda_1 \int_s^t \left(C \frac{\tau \|g\|_{L_0^2(\tau)}^2}{\nu^3 \lambda_1^{1/2}} - 1 \right) \|w(\xi)\|_2^2 d\xi.$$

Assuming that $\|g\|_{L_0^2(\tau)}^2$ is sufficiently small, we can ensure that $C\tau\|g\|_{L_0^2(\tau)}^2 < \nu^3 \lambda_1^{1/2}$ giving us that

$$(17) \quad \|w(t)\|_2^2 - \|w(s)\|_2^2 \leq -M \int_s^t \|w(\xi)\|_2^2 d\xi$$

for $M := \nu \lambda_1 \left(1 - C \frac{\tau \|g\|_{L_0^2(\tau)}^2}{\nu^3 \lambda_1^{1/2}} \right) > 0$. Thus, after applying Gronwall's inequality, we have that

$$\|w(t)\|_2^2 \leq \|w(s)\|_2^2 e^{M(s-t)}.$$

In particular, for t fixed and $s \rightarrow -\infty$, $\|w(t)\|_2 \rightarrow 0$. This completes the proof of the following theorem.

Theorem 3.5. *Let g be translationally bounded in $L_{loc}^2(\mathbb{R}, L^2)$. Assume that $\|g\|_{L_0^2(\tau)}$ is sufficiently small, then the weak pullback attractor for (2) is a single point,*

$$\mathcal{A}_w(t) = \{v(t)\}$$

for some complete, bounded, strong solution to (2).

Again, if we let $\tau := (\nu \lambda_1)^{-1}$, then (16) simplifies to

$$(18) \quad \|w(t)\|_2^2 - \|w(s)\|_2^2 \leq \nu \lambda_1 \int_s^t (CG^2 - 1) \|w(\xi)\|_2^2 d\xi.$$

So, we can restate (3.5) once again in terms of the 3D Grashof constant.

Corollary 3.6. *Let g be translationally bounded in $L_{loc}^2(\mathbb{R}, L^2)$. Assume that the Grashof number G given by*

$$G = \frac{\|g\|_{L_b^2}}{\nu^2 \lambda_1^{3/4}}$$

is sufficiently small, then the weak pullback attractor for (2) is a single point,

$$\mathcal{A}_w(t) = \{v(t)\}$$

for some complete, bounded, strong solution to (2).

3.3. Periodic Force. The existence of a unique periodic solution to the 3D Navier-Stokes equations is a remarkable consequence of this Theorem. To begin, let the force f in (1) be periodic in $L_{loc}^2(\mathbb{R}, L^2)$ with period ρ . Then, the projected force g in (2) is also periodic with period ρ . A straightforward argument shows that g is translationally bounded. Thus, by Theorem 3.5, if g is sufficiently small, there exists a unique, strong solution w to (2). We will show that w is, in fact, periodic.

Theorem 3.7. *Let g be periodic in $L_{loc}^2(\mathbb{R}, L^2)$ with period ρ . Assume that g is sufficiently small. Then, there exists a unique, periodic, strong solution w to (2). In particular, w has period ρ .*

Proof. Due to Theorem 3.5, we only must show that w has period ρ . To this end, note that w satisfies the equation

$$(19) \quad \frac{d}{dt}w(t) - \nu\Delta w(t) + P_\sigma(w \cdot \nabla w)(t) = g(t).$$

Then, of course, w satisfies

$$\frac{d}{dt}w(t + \rho) - \nu\Delta w(t + \rho) + P_\sigma(w \cdot \nabla w)(t + \rho) = g(t + \rho).$$

But, $g(t + \rho) = g(t)$. So, $w(\cdot + \rho)$ also satisfies (19). By uniqueness, $w(t + \rho) = w(t)$. \square

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